Published by Institute of Physics Publishing for SISSA/ISAS
Received: *June 12, 2002*
Accepted: *June 18, 2002*

Two-loop self-dual Euler-Heisenberg lagrangians 2. Imaginary part and Borel analysis

Gerald V. Dunne
Department of Physics University of Connecticut
Storrs CT 06269, USA
E-mail: dunne@phys.uconn.edu

Christian Schubert
Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo
Apdo. Postal 2-82, C.P. 58040, Morelia, Michoacán, México, and
Center for Mathematical Physics, Mathematics Department
Boston University, Boston, MA 02215, USA and
California Institute for Physics and Astrophysics
366 Cambridge Ave., Palo Alto, CA 94306, USA
E-mail: schubert@itzel.ifm.umich.mx

Abstract: We analyze the structure of the imaginary part of the two-loop Euler-Heisenberg QED effective lagrangian for a constant self-dual background. The novel feature of the two-loop result, compared to one-loop, is that the prefactor of each exponential (instanton) term in the imaginary part has itself an asymptotic expansion. We also perform a high-precision test of Borel summation techniques applied to the weak-field expansion, and find that the Borel dispersion relations reproduce the full prefactor of the leading imaginary contribution.

Keywords: Renormalization Regularization and Renormalons, Nonperturbative Effects, Electromagnetic Processes and Properties.

JHEP06(2002)042

Contents

1. Introduction: imaginary part of effective lagrangians **1**

2. Weak-field and strong-field limits (scalar QED) **6**
2.1 One loop 7
2.2 Two loop 11

3. Borel analysis **13**
3.1 Borel analysis of one loop case (scalar QED) 15
3.2 Borel analysis of two-loop case (scalar QED) 17

4. Conclusions **20**

A. Asymptotic expansion of two-loop coefficients in scalar QED **22**

1. Introduction: imaginary part of effective lagrangians

The one-loop Euler-Heisenberg effective lagrangians, for spinor QED [1, 2, 3] and scalar QED [3], in a constant background electromagnetic field, have the following well-known integral representations:

$$\mathcal{L}^{(1)}_{\rm spin}(a,b) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\left[\frac{e^2ab}{\tanh(eaT)\tan(ebT)} - \frac{1}{T^2} - \frac{e^2}{3}(a^2-b^2)\right] \quad (1.1)$$

$$\mathcal{L}^{(1)}_{\rm scal}(a,b) = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\left[\frac{e^2ab}{\sinh(eaT)\sin(ebT)} - \frac{1}{T^2} + \frac{e^2}{6}(a^2-b^2)\right]. \quad (1.2)$$

Here T denotes the (euclidean) proper-time of the loop fermion or scalar, and a, b are related to the two invariants of the Maxwell field by $a^2-b^2 = B^2-E^2, ab = \mathbf{E}\cdot\mathbf{B}$. These lagrangians have been renormalized [1, 2, 3] on-shell by subtracting, under the integral, the terms of zeroeth and second order in a, b.

The effective lagrangians (1.1) and (1.2) are real for a purely magnetic field, while in the presence of an electric field there is an imaginary (absorptive) part, indicating the process of electron-positron (resp. scalar-antiscalar) pair creation by the field. The vacuum persistence amplitude is $e^{i\mathcal{L}}$, so the probability of pair production is approximately $2\,\mathrm{Im}\,\mathcal{L}$ [3]. For example, in the case of a purely electric field, E, the effective lagrangians (1.1) and (1.2) have imaginary part given by:

$$\mathrm{Im}\,\mathcal{L}^{(1)}_{\mathrm{spin}}(E) = \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{\pi k}{\beta}\right] \tag{1.3}$$

$$\mathrm{Im}\,\mathcal{L}^{(1)}_{\mathrm{scal}}(E) = -\frac{m^4}{16\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \exp\left[-\frac{\pi k}{\beta}\right] \tag{1.4}$$

where $\beta = eE/m^2$. These expressions are clearly non-perturbative in terms of the field and coupling.

The physical interpretation of these imaginary parts (1.3) and (1.4) of the effective lagrangians is that the coefficient of the k-th exponential can be directly identified with the rate for the coherent production of k pairs by the field [3, 4]. This physical meaning of the individual terms of these series follows most directly from the alternative representation due to Nikishov [4, 5],

$$\begin{aligned} VT\frac{2}{\hbar}\,\mathrm{Im}\,\mathcal{L}^{(1)}(E) &= \mp\sum_r \int \frac{V\,d^3p}{(2\pi\hbar)^3}\ln(1\mp\bar{n}_p)\,, \\ \bar{n}_p &= \exp\left(-\pi\frac{m^2+p_\perp^2}{eE}\right). \end{aligned} \tag{1.5}$$

Here $\bar{n}_p$ is the mean number of pairs produced by the field in the state with given momentum p and spin projection r, the $\mp$ refers to the spinor/scalar cases respectively, and $p_\perp$ is the momentum transverse to the field. An expansion of the logarithm in $\bar{n}_p$ and term-by-term integration leads back to Schwinger's formulas (1.3) and (1.4). Thus the leading term in this expansion can be interpreted as the mean number $\bar{n}_p$ of pairs in the unit 4-volume VT, while the higher ($k \geq 2$) terms describe the coherent creation of k pairs. This also explains the physical origin of the alternating signs in the scalar case (1.4).

The exponential suppression factors in (1.3) and (1.4) are very small for the field strengths which are presently possible for a macroscopic field in the laboratory. While positron production has recently been observed in an experiment involving electrons traversing the focus of a terawatt laser [6], in this experiment the electric field strength is such that it is on the border-line between the perturbative and nonperturbative regimes, and so the positron production can be explained by multiphoton Compton scattering [6], rather than in terms of nonperturbative pair creation [7]. This phenomenon has been analyzed in detail in [8], along with the prospects of observing nonperturbative pair creation using an X-ray free-electron laser which produces higher fields than those obtained by an optical laser. We mention also a recent analysis [9] which considers nonequilibrium and back-reaction effects.

JHEP06(2002)042

From a computational perspective, the imaginary parts of the effective lagrangian can be computed in several ways. At one-loop, the most direct way is to consider the analytic properties of the integral representations in (1.1) and (1.2). However, this approach is quite diffcult at the two-loop level for an electric field background, because the corresponding integral representations are double integrals, with much more complicated integrands (see discussion below). An alternative approach is to exploit the well-known relation between the large-order divergence of perturbation theory and non-perturbative physics [10, 11, 12]. For an electric field background, the weak-field expansion of the effective lagrangian is a non-alternating divergent series, which signals the presence of an exponentially small imaginary part of the effective lagrangian. Knowledge of the rate of divergence of the perturbation series can be used to deduce the exponentially small imaginary part of the effective lagrangian, and therefore the pair production rate.

This correspondence, between the divergent perturbative weak-field expansion and the non-perturbative imaginary part, is well understood at the one-loop level. For example, the weak-field expansions of the one-loop effective lagrangians (1.1) and (1.2), specialized to the constant electric field case, are:

$$\mathcal{L}^{(1)}_{\rm spin}(E) = -\frac{2m^4}{\pi^2}\left(\frac{eE}{m^2}\right)^4 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left(\frac{eE}{m^2}\right)^{2n} \tag{1.6}$$

$$\mathcal{L}^{(1)}_{\rm scal}(E) = \frac{m^4}{\pi^2}\left(\frac{eE}{m^2}\right)^4 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}(2^{-2n-3}-1) \mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left(\frac{eE}{m^2}\right)^{2n} \tag{1.7}$$

where the coefficients involve the Bernoulli numbers $\mathcal{B}_{2n}$, which alternate in sign and diverge factorially fast in magnitude [13, 14]:

$$\mathcal{B}_{2n} = (-1)^{n+1} 2 \frac{(2n)!}{(2\pi)^{2n}} \zeta(2n) \tag{1.8}$$

Here $\zeta(n)$ denotes the Riemann zeta function [13]

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \tag{1.9}$$

which is exponentially close to 1 for large-N. Also noting that [13]

$$(1-2^{1-n})\zeta(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \tag{1.10}$$

we see that the weak-field expansion coefficients in (1.6) and (1.7) are

$$a_n^{(1){\rm spin}} = \frac{2m^4}{\pi^2}\left(\frac{eE}{m^2}\right)^4 \frac{\Gamma(2n+2)}{8} \sum_{k=1}^{\infty} \frac{1}{(\pi k)^{2n+4}} \tag{1.11}$$

$$a_n^{(1){\rm scal}} = \frac{m^4}{\pi^2}\left(\frac{eE}{m^2}\right)^4 \frac{\Gamma(2n+2)}{8} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(\pi k)^{2n+4}} \tag{1.12}$$

There is a precise one-to-one correspondence (as will be reviewed in section 3) between the terms in these large-order behaviors of the weak-field expansion coefficients, and the terms in the non-perturbative expressions (1.3) and (1.4) for the imaginary parts of the one-loop effective lagrangians.

At the two-loop level, the situation is even more interesting. The first radiative corrections to the Euler-Heisenberg lagrangians (1.1) and (1.2), describing the effect of an additional photon exchange in the loop, were first studied in the seventies by Ritus [15, 16, 17]. Using the exact spinor and scalar propagators in a constant field found by Fock [18] and Schwinger [3], and a proper-time cutoff as the UV regulator, Ritus obtained the two-loop contribution $\mathcal{L}^{(2)}$ in terms of a certain two-parameter integral. This integral representation was then used in [19] for deriving, by an analysis of the analyticity properties of the integrand, a representation for the imaginary part $\mathrm{Im}\,\mathcal{L}^{(2)}_{\mathrm{spin}}$, analogous to Schwinger's one-loop formula (1.3). Adding together the one-loop and the two-loop contributions, the imaginary part reads, in the purely electric case for spinor QED,

$$\mathrm{Im}\,\mathcal{L}^{(1)}_{\mathrm{spin}}(E)+\mathrm{Im}\,\mathcal{L}^{(2)}_{\mathrm{spin}}(E)=\frac{m^4}{8\pi^3}\beta^2\sum_{k=1}^{\infty}\left[\frac{1}{k^2}+\alpha\pi K_k(\beta)\right]\exp\left[-\frac{\pi k}{\beta}\right] \tag{1.13}$$

where $\alpha=\frac{e^2}{4\pi}$ is the fine-structure constant. The coefficient functions $K_k(\beta)$ appearing here were not obtained explicitly by [19]. However, it was shown that they have small β expansions of the following form:

$$\begin{aligned} K_k(\beta) &= -\frac{c_k}{\sqrt{\beta}}+1+\mathrm{O}(\sqrt{\beta}) \\ c_1 &= 0\,, \qquad c_k=\frac{1}{2\sqrt{k}}\sum_{l=1}^{k-1}\frac{1}{\sqrt{l(k-l)}}\,, \qquad k\geq 2 \end{aligned} \tag{1.14}$$

Note that, for $k\geq 2$, these expansions start with terms that are singular in the limit of vanishing field $\beta\to 0$, which seems to be at variance with the fact that these coefficients have a direct physical meaning. In [19] a physically intuitive solution was offered to this dilemma. Its basic assumption is that, if one would take into account all contributions from even higher loop orders to the prefactor of the k-th exponential, then one would find them to exponentiate in the following way,

$$\left[\frac{1}{k^2}+\alpha\pi K_k\left(\frac{eE}{m^2}\right)+\cdots\right]\exp\left[-\frac{k\pi m^2}{eE}\right]=\frac{1}{k^2}\exp\left[-\frac{k\pi m_*^2(k,E)}{eE}\right] \tag{1.15}$$

Thus, it should be possible to absorb their effect completely into a field-dependent shift of the electron mass. Using just the lowest order coefficients in the small — β expansion of $K_k(\beta)$, those given explicitly in (1.14), this mass shift reads

$$m_*(k,E)=m+\frac{1}{2}\alpha k c_k\sqrt{eE}-\frac{1}{2}\alpha k eE/m \tag{1.16}$$

As shown in [20, 19] these contributions to the mass shift have a simple meaning in the coherent tunneling picture: The negative term can be interpreted as the total Coulomb energy of attraction between opposite charges in a coherent group; the positive one, which is present only in the case $k\geq 2$, represents the energy of repulsion between like charges.

It is important to note that this interpretation of the mass shift requires the mass m on the right hand side of (1.16) to be the *physical* renormalized electron mass of the vacuum theory. Only in this case the expansion of $K_k(\beta)$ has the form indicated in eqs. (1.14). It is an interesting side result of the Lebedev-Ritus analysis that the physical electron mass can be recognized from an inspection of the two-loop effective lagrangian alone, without ever considering the one-loop electron mass operator.

Clearly, it would be of interest to understand in greater detail this prefactor series $K_k(\beta)$. It is not even clear from the work of Lebedev and Ritus whether or not this is a convergent expansion. In an effort to learn more about this prefactor, the imaginary part of the two-loop effective lagrangian for a constant electric field background was studied using Borel techniques [21]. However, since no closed-form is known for the two-loop weak-field expansion coefficients in the case of a background constant electric field, this analysis was necessarily numerical. The leading, and first sub-leading, growth rates of the expansion coefficients were deduced numerically from the first 15 coefficients in the spinor QED case, using a brute-force expansion of the integral representation (a double integral) of the effective lagrangian. This was sufficient to indicate that there are power-law corrections to the leading factorial growth rate, in contrast to the one-loop results (1.11) and (1.12) which only have *exponential* corrections to the leading growth rate. Using Borel dispersion relations, this indicates the presence of a prefactor series, as in the Ledebev-Ritus result (1.13). In this way, a numerical estimate was obtained for the next term in the expansion of the prefactor $K_1(\beta)$ of the leading exponential term ($k = 1$). This also provided an independent confirmation (albeit numerical) that the expansion proceeds in powers of $\sqrt{\beta}$. However, it was prohibitively difficult to go to much higher order, and so it was only possible to deduce one more term in the prefactor expansion.

However, the case of a constant electric field background is not the simplest case one can study in this context. As shown in part I [22], for a constant self-dual euclidean background, the two-loop effective lagrangians, for both spinor and scalar QED, simplify dramatically, to such a degree that there are simple closed-form expressions in terms of the digamma function [23]. Such a constant background field satisfies

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{1.17}$$

This has the consequence that

$$F^2 = -f^2\mathbb{1} \tag{1.18}$$

In Minkowski space, the self-duality condition (1.17) requires either $\mathbf{E}$ or $\mathbf{B}$ to be complex. This does not imply that such backgrounds are devoid of physical meaning; rather, the effective action in such a background should be understood in terms of helicity projections [24, 22], and contains information on the photon amplitudes with all equal helicities (see part I).

As shown in part I, for such a self-dual background, the two-loop spinor QED and scalar QED renormalized effective lagrangians are:

$$\mathcal{L}_{\rm spin}^{(2)(SD)}(\kappa) = -2\alpha\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[3\xi^2(\kappa) - \xi'(\kappa)\right] \tag{1.19}$$

$$\mathcal{L}_{\rm scal}^{(2)(SD)}(\kappa) = \alpha\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[\frac{3}{2}\xi^2(\kappa) - \xi'(\kappa)\right] \tag{1.20}$$

Here we have defined the convenient dimensionless parameter

$$\kappa \equiv \frac{m^2}{2e\sqrt{f^2}} \tag{1.21}$$

as well as the important function

$$\xi(x) \equiv= -x\left(\psi(x) - \ln(x) + \frac{1}{2x}\right) \tag{1.22}$$

with ψ being the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$ [13, 14].

In [22], the parameter κ, defined in (1.21), was taken to be real. In Minkowski space, this corresponds to taking the magnetic field B to be real, and the electric field E to be imaginary — so we refer to this case as the self-dual 'magnetic' case. In this case, the two-loop effective lagrangians (1.19) and (1.20) are real. However, we could also analytically continue $\kappa \to i\kappa$, which in the Minkowski language corresponds to taking E to be real and B to be imaginary. Thus, we refer to this case as the self-dual 'electric' case. For this background, with κ imaginary, the two-loop effective lagrangians (1.19) and (1.20) acquire exponentially small imaginary parts. As we shall see, these imaginary parts share many of the properties of the imaginary parts in (1.3) and (1.4), for the case of a constant electric field background.

The obvious advantage of the self-dual case is that since there are simple closed-form expressions for the two-loop effective lagrangians (1.19) and (1.20), it is possible to study the imaginary part in much greater detail. In this paper, we analyze these imaginary parts, and compare the direct approach with the Borel dispersion relation approach. We are able to find the entire prefactor series for the imaginary parts. Since the spinor QED and scalar QED cases are so similar for a self-dual background, to avoid repetition, we concentrate on just one of them — the scalar QED case. In section 2 we present the details of the weak- and strong-field expansions of the one- and two-loop effective lagrangians for scalar QED in a constant self-dual background. In section 3 we apply Borel summation techniques to the weak-field expansion, and use Borel dispersion relations to compute the imaginary part of the effective lagrangian when the field strength is analytically continued $\kappa \to i\kappa$. Section 4 contains some concluding remarks, and appendix A contains the details of the derivation of the large-order behavior of the two-loop weak-field expansion coefficients.

2. Weak-field and strong-field limits (scalar QED)

In this section we discuss the weak- and strong-field limits of the on-shell renormalized one- and two-loop effective actions for scalar QED in a constant euclidean self-dual background.

We begin by recalling (1.21) that $\kappa = \frac{m^2}{2e\sqrt{f^2}}$, where f is defined in (1.18), so that "weak-field" means large κ, and "strong-field" means small κ. We first review the one-loop case, and then turn to the two-loop case.

2.1 One loop

In the self-dual ('SD') case, the integral representation (1.2) for the renormalized one-loop effective lagrangian becomes

$$\mathcal{L}^{(1)(SD)}_{\rm scal}(\kappa) = \frac{m^4}{(4\pi)^2}\frac{1}{4\kappa^2}\int_0^\infty \frac{dt}{t^3} e^{-2\kappa t}\left[\frac{t^2}{\sinh^2(t)} - 1 + \frac{t^2}{3}\right] \tag{2.1}$$

The weak-field (large κ) expansion of this proper-time integral representation can be derived using the Taylor expansion [13, 14]

$$\left(\frac{x}{\sinh(x)}\right)^2 = -\sum_{k=0}^\infty \frac{(2k-1)2^{2k}}{(2k)!}\mathcal{B}_{2k}x^{2k} \tag{2.2}$$

This leads to the weak-field expansion

$$\mathcal{L}^{(1)(SD)}_{\rm scal}(\kappa) = \frac{m^4}{(4\pi)^2}\sum_{n=2}^\infty c_n^{(1)}\frac{1}{\kappa^{2n}} \tag{2.3}$$

where the expansion coefficients are (for $n \geq 2$):

$$c_n^{(1)} = -\frac{\mathcal{B}_{2n}}{2n(2n-2)} \tag{2.4}$$

The leading term in the weak-field expansion (2.3) is

$$\mathcal{L}^{(1)(SD)}_{\rm scal}(\kappa) \sim \frac{m^4}{(4\pi)^2}\frac{1}{240\kappa^4} \tag{2.5}$$

To derive the strong-field (small κ) expansion, it is convenient to express (2.1) in terms of special functions as

$$\mathcal{L}^{(1)(SD)}_{\rm scal}(\kappa) = \frac{m^4}{(4\pi)^2}\frac{1}{\kappa^2}\left[-\frac{1}{12}\ln\kappa + \zeta'(-1) + \Xi(\kappa)\right] \tag{2.6}$$

where the function $\Xi(\kappa)$ is defined as [25]

$$\Xi(\kappa) \equiv -\kappa\ln\Gamma(\kappa) + \frac{\kappa^2}{2}\ln\kappa - \frac{\kappa^2}{4} - \frac{\kappa}{2} + \int_0^\kappa dy\ln\Gamma(y) \tag{2.7}$$

and $\zeta'(-1) \approx -0.16542$. Note that this function $\Xi(\kappa)$ is simply related to the function $\xi(\kappa)$, defined in (1.22), which appears in the two-loop expressions (1.19) and (1.20) by $\Xi'(\kappa) = \xi(\kappa)$.

Thus, using the Taylor expansion of $\ln\Gamma(x)$,

$$\ln\Gamma(x) = -\ln x - \gamma x + \sum_{n=2}^\infty \frac{(-1)^n\zeta(n)}{n}x^n \tag{2.8}$$

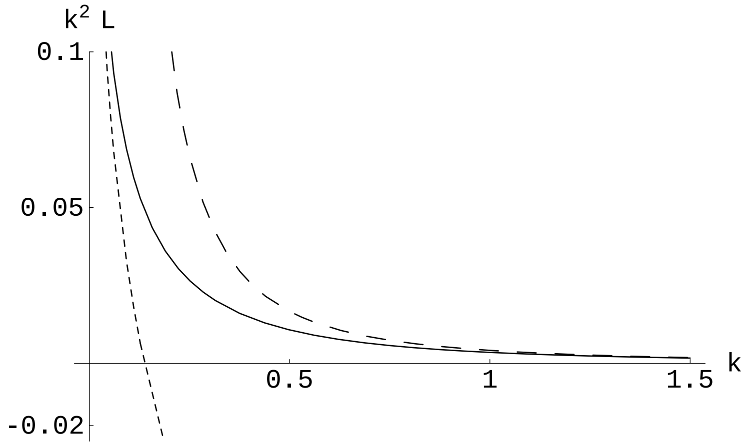

Figure 1: This plot shows the exact one-loop scalar QED effective lagrangian (2.1) [solid curve] in comparison with the leading weak-field (large κ) [long-dashed curve] and strong-field (small κ) [short-dashed curve] contributions (2.5) and (2.10), respectively. We have multiplied through each function by a common factor of $(4\pi)^2\kappa^2/m^4$ for ease of presentation.

the strong-field (small κ) expansion of the one-loop effective action (2.1) is

$$\begin{aligned}\mathcal{L}^{(1)(SD)}_{\rm scal}(\kappa) &= \frac{m^4}{(4\pi)^2\kappa^2}\times \\ &\times\left[\left(-\frac{1}{12}+\frac{\kappa^2}{2}\right)\ln\kappa+\zeta'(-1)+\frac{\kappa}{2}+\left(\frac{\gamma}{2}-\frac{1}{4}\right)\kappa^2-\sum_{n=2}^{\infty}\frac{(-1)^n\zeta(n)}{(n+1)}\kappa^{n+1}\right]\end{aligned} \tag{2.9}$$

The leading strong-field behavior in (2.9) is

$$\mathcal{L}^{(1)(SD)}_{\rm scal}(\kappa)\sim\frac{m^4}{(4\pi)^2}\frac{1}{\kappa^2}\left[-\frac{1}{12}\ln\kappa+\zeta'(-1)\right] \tag{2.10}$$

Figure 1 shows a plot of (κ^2 times) the exact one-loop effective action (1.12), compared with (κ^2 times) the leading weak- and strong-field behaviors. (We multiply by the common factor of κ^2 for clarity of presentation over a reasonable range of κ). From figure 1, it is clear that the leading behaviors (2.5) and (2.10) correctly capture the extremes, but fail to connect well in the intermediate region. (See [26] for a simple approximate interpolation method to connect the weak- and strong-field extremes).

From (2.4) and (1.8), it is clear that the weak-field expansion (2.3) is a divergent expansion. The first 15 expansion coefficients $c_n^{(1)}$ are tabulated in table 1, together with their magnitudes. Notice that the coefficients alternate in sign (here κ is real), and their magnitude eventually grows very fast. In fact, the leading growth rate of the coefficients at large-N is factorial:

$$c_n^{(1)}\sim 2\frac{(-1)^n}{(2\pi)^{2n}}\Gamma(2n-1)\left[1+O\left(\frac{1}{n}\right)\right] \tag{2.11}$$

Figure 2 shows a plot comparing the exact one-loop result (2.1) with successive truncations of the weak-field series expansion in (2.3). (Note once again that these plots have

n	$c_n^{(1)}$	$N[\lvert c_n^{(1)}\rvert]$	$c_n^{(2)}$	$N[\lvert c_n^{(2)}\rvert]$
2	$\frac{1}{240}$	0.00416667	$\frac{3}{128\pi^2}$	0.00237472
3	$-\frac{1}{1008}$	0.000992063	$-\frac{13}{1920\pi^2}$	0.000686029
4	$\frac{1}{1440}$	0.000694444	$\frac{67}{12800\pi^2}$	0.000530353
5	$-\frac{1}{1056}$	0.00094697	$\frac{-611}{80640\pi^2}$	0.000767699
6	$\frac{691}{327600}$	0.00210928	$\frac{3269351}{186278400\pi^2}$	0.00177828
7	$-\frac{1}{144}$	0.00694444	$\frac{-684779}{11531520\pi^2}$	0.00601678
8	$\frac{3617}{114240}$	0.0316614	$\frac{42467137}{153753600\pi^2}$	0.0279852
9	$-\frac{43867}{229824}$	0.190872	$\frac{-2783806241}{1646701056\pi^2}$	0.171287
10	$\frac{174611}{118800}$	1.46979	$\frac{308416614839}{23412280320\pi^2}$	1.33473
11	$-\frac{77683}{5520}$	14.073	$\frac{-14229172307981}{111740428800\pi^2}$	12.9024
12	$\frac{236364091}{1441440}$	163.978	$\frac{20984465589542501429}{14032363048704000\pi^2}$	151.519
13	$-\frac{657931}{288}$	2284.48	$\frac{-7699261058623757}{367147123200\pi^2}$	2124.76
14	$\frac{3392780147}{90480}$	37497.6	$\frac{889545454541748900227}{25700298624000\pi^2}$	35069.6
15	$-\frac{1723168255201}{2406096}$	716168	$\frac{-114354778628165307771811}{17216630048217600\pi^2}$	672987
16	$\frac{7709321041217}{489600}$	1.57462×10^7	$\frac{460780386050723601316078067}{3142221310964736000\pi^2}$	1.48579×10^7

Table 1: The first 15 one- and two-loop weak-field expansion coefficients, $c_n^{(1)}$ and $c_n^{(2)}$ appearing in (2.4) and (2.16), together with their numerical magnitudes. Note that in each case the expansion coefficients alternate in sign and grow factorially fast in magnitude. Furthermore, the leading growth rate at one- and two-loop is the same.

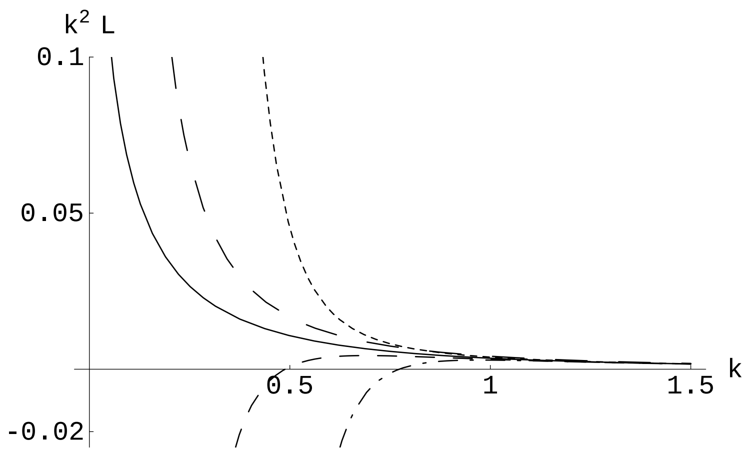

Figure 2: This plot shows the exact one-loop scalar QED effective lagrangian (2.1) [solid curve] in comparison with successive partial sums of the weak-field (large κ) expansion (2.3). The partial sum is shown for one term [long-dashed-curve], two terms [medium-dashed curve], three terms [short-dashed curve], and four terms [dot-dash curve]. We have multiplied through each function by a common factor of $(4\pi)^2\kappa^2/m^4$ for ease of presentation.

been multiplied by a common factor of κ^2). We see that the successive truncations form an envelope around the exact result. This is typical behavior for an alternating asymptotic expansion.

n	$\kappa = 0.5$	$\kappa = 1$	$\kappa = 1.5$	$\kappa = 2$
2	0.0666667	0.00416667	0.000823045	0.000260417
3	0.0634921	0.000992063	0.0000870947	0.000015501
4	0.177778	0.000694444	0.0000270961	2.71267×10^{-6}
5	0.969697	0.00094697	0.0000164219	9.24775×10^{-7}
6	8.63961	0.00210928	0.0000162569	5.14961×10^{-7}
7	113.778	0.00694444	0.0000237881	4.23855×10^{-7}
8	2074.96	0.0316614	0.0000482026	4.83115×10^{-7}
9	50036	0.190872	0.000129152	7.28119×10^{-7}
10	1.54119×10^{6}	1.46979	0.000442008	1.4017×10^{-6}
11	5.90265×10^{7}	14.073	0.00188096	3.35527×10^{-6}
$n_{\min}$	3	4	6	7
$N_0 = \frac{1}{2} + \pi\kappa \exp(\frac{1}{4\pi\kappa})$	2.34179	3.90181	5.46914	7.03823
optimal partial sum	0.0666667	0.0031746	0.000746625	0.000247219
exact value	0.0429404	0.00351739	0.000753997	0.000247005
% error	55.2539	9.74548	0.977739	0.0865725

Table 2: This table shows the magnitudes of the terms $c_n^{(1)}/\kappa^{2n}$ in the one-loop weak-field expansion (2.3), for four different values of κ. In each column, note that the terms decrease in magnitude and then increase in magnitude. The order of the minimum term is $n_{\min}$, and this is compared with the estimate N_0 of the order of the minimum term given in (2.14). The optimal partial sum is the partial sum up to the $(n_{\min} - 1)^{th}$ term. This is compared with the exact value and the fractional error is shown as a percentage.

To probe more deeply the asymptotic nature of the weak-field expansion we consider the optimal asymptotic approximation [27, 28] given by truncating the series in such a way that the error is minimized. (More sophisticated estimation techniques can be applied to the series, but the simplest one is sufficient here). To find the optimal truncation point of an asymptotic series

$$f(x) = \sum_n (-1)^n c_n x^n \tag{2.12}$$

we evaluate the terms $c_n x^n$ and find the minimum value, say $c_{N_0} x^{N_0}$. Then the optimal asymptotic approximation consists of keeping terms up to $n = N_0 - 1$. In this way the N_0^{th} term, which is a measure of the truncation error, is minimized. Clearly, the truncation point depends on the value of expansion parameter x.

For the specific case of the weak field expansion (2.3), we define the remainder as

$$\begin{aligned} R_N(\kappa) &= -\frac{m^4}{(4\pi)^2} \frac{\mathcal{B}_{2N}}{2N(2N-2)} \frac{1}{\kappa^{2N}} \\ &\sim -\frac{m^4}{(4\pi)^2} 2\Gamma(2N-1) e^{-N \ln(2\pi\kappa)^2} \end{aligned} \tag{2.13}$$

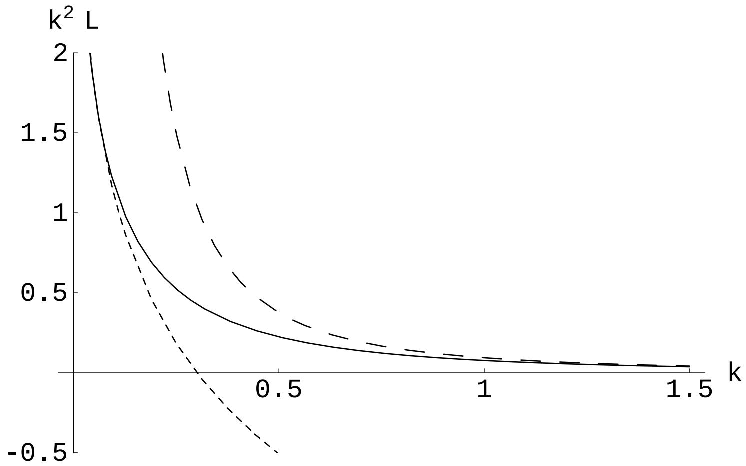

Figure 3: This plot shows the exact two-loop scalar QED effective lagrangian (1.20) [solid curve] in comparison with the leading weak-field (large κ) [long-dashed curve] and strong-field (small κ) [short-dashed curve] contributions (2.18) and (2.21), respectively. We have multiplied through each function by a common factor of $(4\pi)^2\kappa^2/(\alpha\pi m^4)$ for ease of presentation.

JHEP06(2002)042

To find the optimal truncation point N_0, we minimize $|R_N(\kappa)|$ with respect to N. The magnitude of this remainder is minimized when $\ln(2\pi\kappa) = \psi(2N-1) \approx \ln(2N-1) - 1/(2(2N-1)) + \cdots$. Thus, a simple estimate of the point at which the remainder is minimized is given by

$$N_0 \approx \frac{1}{2} + \pi\kappa \exp\left(\frac{1}{4\pi\kappa}\right) \tag{2.14}$$

In table 2 we compare this optimal asymptotic approximation with the exact answer, for various values of κ. Note that for $\kappa < 1$ (i.e., "strong field", where $f > \frac{m^2}{2e}$), the optimal truncation involves keeping just the first term in the weak-field expansion. Keeping more terms in the series makes the series estimate worse.

2.2 Two loop

At two-loop, for scalar QED in a constant euclidean selfdual background, the on-shell renormalized effective lagrangian is given by (1.20). The weak-field (large κ) expansion of this result is

$$\mathcal{L}^{(2)(SD)}_{\rm scal}(\kappa) = \alpha\pi \frac{m^4}{(4\pi)^2} \sum_{n=2}^{\infty} c_n^{(2)} \frac{1}{\kappa^{2n}} \tag{2.15}$$

where the two-loop expansion coefficients are (for $n \geq 2$):

$$c_n^{(2)} = \frac{1}{(2\pi)^2}\left\{\frac{2n-3}{2n-2}\mathcal{B}_{2n-2} + \frac{3}{2}\sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}}{2k}\frac{\mathcal{B}_{2n-2k}}{(2n-2k)}\right\} \tag{2.16}$$

This weak-field expansion follows directly from (1.20) and (1.22), together with the asymptotic (large x) expansion of the digamma function [13, 14]:

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{k=1}^{\infty}\frac{\mathcal{B}_{2k}}{2kx^{2k}} \tag{2.17}$$

The leading term in the weak-field expansion (2.15) is

$$\mathcal{L}_{\rm scal}^{(2)(SD)}(\kappa) \sim \alpha\pi \frac{m^4}{(4\pi)^2} \frac{3}{128\pi^2\kappa^4} \tag{2.18}$$

The strong-field (small κ) expansion follows directly from (1.20) and (1.22), together with the small argument expansion of the digamma function [13, 14]:

$$\psi(x) \sim -\frac{1}{x} - \gamma + \sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-1} \tag{2.19}$$

This leads to:

$$\begin{aligned}
&\mathcal{L}_{\rm scal}^{(2)(SD)}(\kappa) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.20)\\
&= \frac{m^4\alpha}{(4\pi)^3}\frac{1}{\kappa^2}\Bigg[(\gamma+\ln\kappa)\left\{-1+\frac{3}{2}\kappa+\frac{3}{2}\kappa^2(\gamma+\ln\kappa)-3\sum_{n=2}^{\infty}(-1)^n\zeta(n)\kappa^{n+1}\right\}-\frac{5}{8}+\frac{\pi^2}{3}\kappa-\\
&\qquad -\sum_{n=2}^{\infty}(-1)^n\left(\frac{3}{2}\zeta(n)+(n+1)\zeta(n+1)\right)\kappa^n+\frac{3}{2}\sum_{n=4}^{\infty}\sum_{l=2}^{n-2}(-1)^n\zeta(n-l)\zeta(l)\kappa^n\Bigg]
\end{aligned}$$

The leading strong-field behavior is

$$\mathcal{L}_{\rm scal}^{(2)(SD)} \sim \frac{m^4\alpha}{(4\pi)^3}\frac{1}{\kappa^2}\left[-\ln\kappa-\frac{5}{8}-\gamma\right] \tag{2.21}$$

Figure 3 shows a plot of (κ^2 times) the exact expression (1.20), in comparison with (κ^2 times) these leading weak-field and strong-field behaviors, (2.18) and (2.21), respectively. This is completely analogous to the one-loop case (see figure 1). The extreme behavior for large and small κ is well described, but they do not connect well in the intermediate region.

The two-loop weak-field expansion (2.15) is a divergent asymptotic expansion; it is simply based on the asymptotic expansion (2.17) of the digamma function $\psi(\kappa)$. Figure 4 shows a comparison of the exact two-loop expression (1.20) with successive truncations of the weak-field expansion in (2.15) (Once again, a common extra factor of κ^2 is included for clarity of plotting). The pattern is much the same as in the one-loop case shown in figure 2, and is characteristic of an asymptotic expansion.

The two-loop expansion coefficients (2.16) alternate in sign and grow factorially fast in magnitude, as shown in table 1. In fact, the precise leading growth rate is

$$c_n^{(2)} \sim 2\frac{(-1)^n}{(2\pi)^{2n}}\Gamma(2n-1)\left[1+O\left(\frac{1}{n}\right)\right] \tag{2.22}$$

which is exactly the same as the growth rate of the one-loop cofficients $c_n^{(1)}$ in (2.11). This equality between the one-loop and two-loop leading growth rates can even be seen in the first 15 coefficients shown in table 1. This equality will become significant in the following when we discuss the Borel summation properties of the weak-field expansion. It also means that the estimate of the optimal truncation point is exactly the same as in the one-loop case, so we do not repeat the argument.

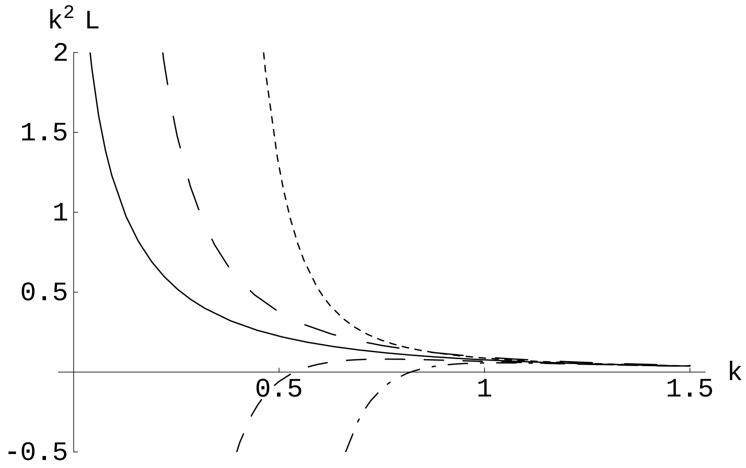

Figure 4: This plot shows the exact two-loop scalar QED effective lagrangian (1.20) [solid curve] in comparison with successive partial sums of the weak-field (large κ) expansion (2.15). The partial sum is shown for one term [long-dashed-curve], two terms [medium-dashed curve], three terms [short-dashed curve], and four terms [dot-dash curve]. We have multiplied through each function by a common factor of $(4\pi)^2\kappa^2/(\alpha\pi m^4)$ for ease of presentation.

3. Borel analysis

The weak-field expansions (2.3) and (2.15) at one- and two-loop, respectively, are divergent asymptotic expansions. In this section we apply the technique of Borel summation to obtain approximate expressions for the corresponding one- and two-loop effective lagrangians. This enables us to test the Borel method with great accuracy because we are in the unusual position of having the explicit closed-form expressions (2.1) and (1.20) for the effective lagrangians themselves, as well as the explicit expressions (2.4) and (2.16) for the expansion coefficients to all orders. We also consider the issue of analytic continuation in the field strength, and compare the Borel results with the exact results.

Strictly speaking, it is, of course, unnecessary to perform a Borel resummation in these cases, since we know the exact closed-form answers. However, such knowledge of the exact answer is rare, and our aim in this section is to probe the power, as well as the limitations, of the Borel approach in a case where a precise comparison can be made.

Note of course that divergent behavior is not a bad thing; it is in fact a generic behavior in perturbation theory, as is illustrated in numerous and diverse examples in both quantum field theory and quantum mechanics [29, 10, 11, 12]. It is well known that knowledge of the divergence rate of high orders of perturbation theory can be used to extract information about non-perturbative decay and tunneling rates, thereby providing a bridge between perturbative and non-perturbative physics.

To begin, we review very briefly some basics of Borel summation [30, 27, 12, 31]. Consider an asymptotic series expansion of some function $f(g)$

$$f(g) \sim \sum_{n=0}^{\infty} a_n g^n \tag{3.1}$$

where $g \to 0^+$ is a small dimensionless perturbation expansion parameter. In an extremely broad range of physics applications [11] one finds that perturbation theory leads not to a convergent series but to a divergent series in which the expansion coefficients a_n have leading large-order behaviour

$$a_n \sim (-1)^n \rho^n \Gamma(\mu n + \nu) \qquad (n \to \infty) \tag{3.2}$$

for some real constants ρ, $\mu > 0$, and ν. When $\rho > 0$, the perturbative expansion coefficients a_n alternate in sign and their magnitude grows factorially, just as in the weak-field expansions studied here: see (2.11) and (2.22). Borel summation is a useful approach to this case of a divergent, but alternating series. Non-alternating series must be treated somewhat differently (see below).

To motivate the Borel approach, consider the classic example : $a_n = (-1)^n \rho^n n!$, and $\rho > 0$. The series (3.1) is clearly divergent for any value of the expansion parameter g. Write

$$\begin{aligned} f(g) &\sim \sum_{n=0}^{\infty} (-1)^n (\rho g)^n \int_0^\infty dt t^n e^{-t} \\ &\sim \frac{1}{\rho g} \int_0^\infty dt \left(\frac{1}{1+t} \right) \exp\left[-\frac{t}{\rho g} \right] \end{aligned} \tag{3.3}$$

where we have formally interchanged the order of summation and integration. The final integral, which is convergent for all $g > 0$, is *defined* to be the sum of the divergent series. To be more precise [30, 27], the formula (3.3) should be read backwards: for $g \to 0^+$, we can use Laplace's method to make an asymptotic expansion of the integral, and we obtain the asymptotic series in (3.1) with expansion coefficients $a_n = (-1)^n \rho^n n!$. This example captures the essence of the Borel method.

For a non-alternating series, such as $a_n = \rho^n n!$, we need $f(-g)$. The particular Borel integral (3.3) is [30, 27] an analytic function of g in the cut g plane: $|\arg(g)| < \pi$. So a dispersion relation (using the discontinuity across the cut along the negative g axis) can be used to *define* the imaginary part of $f(g)$ for negative values of the expansion parameter:

$$\mathrm{Im}\left[f(-g) \right] \sim \frac{\pi}{\rho g} \exp\left[-\frac{1}{\rho g} \right] \tag{3.4}$$

The imaginary contribution (3.4) is non-perturbative (it clearly does not have an expansion in positive powers of g) and has important physical consequences. Note that (3.4) is consistent with a principal parts prescription for the pole that appears on the $s > 0$ axis if we make the formal manipulations as in (3.3):

$$\sum_{n=0}^{\infty} \rho^n n! g^n \sim \frac{1}{\rho g} \int_0^\infty dt \left(\frac{1}{1-t} \right) \exp\left[-\frac{t}{\rho g} \right] \tag{3.5}$$

Similar formal arguments can be applied to the case when the expansion coefficients have leading behaviour (3.2). Then the leading Borel approximation is

$$f(g) \sim \frac{1}{\mu} \int_0^\infty \frac{dt}{t} \left(\frac{1}{1+t} \right) \left(\frac{t}{\rho g} \right)^{\nu/\mu} \exp\left[-\left(\frac{t}{\rho g} \right)^{1/\mu} \right] \tag{3.6}$$

For the corresponding non-alternating case, when g is negative, the leading imaginary contribution is

$$\mathrm{Im}\left[f(-g)\right] \sim \frac{\pi}{\mu}\left(\frac{1}{\rho g}\right)^{\nu/\mu} \exp\left[-\left(\frac{1}{\rho g}\right)^{1/\mu}\right] \tag{3.7}$$

Note the separate meanings of the parameters ρ, μ and ν that appear in the formula (3.2) for the leading large-order growth of the expansion coefficients. The constant ρ clearly combines with g as an effective expansion parameter. The power of the exponent in (3.6) and (3.7) is determined by μ, while the power of the prefactor in (3.6) (3.7) is determined by the ratio ν/μ.

To illustrate this correspondence, between the perturbative expansion coefficients (3.2) and the imaginary part (3.7), it is now a straightforward exercise to check that for the one-loop electric field background, the weak-field expansion coefficients in (1.11) and (1.12) correspond to the imaginary part of the respective effective lagrangians given in (1.3) and (1.4).

These formulas (3.6) and (3.7) are somewhat formal, as they are based on assumed analyticity properties of the function $f(g)$. These can be rigorously proved when the expansion coefficients are precisely given by the gamma function form in (3.2). But things are less clear when this expression gives just the *approximate* large order behavior of the coefficients. The Borel dispersion relations could be complicated by the appearance of additional poles and/or cuts in the complex g plane, signalling new physics, such as, for example, renormalons [31, 32]. Also, the expression (3.7) assumes a principal parts prescription for the poles. Our analysis below will show that these assumptions are justified for the one- and two-loop effective lagrangians (2.1) and (1.20) for scalar QED in a constant euclidean self-dual background. The same is true for the spinor case, but we do not repeat the analysis here: it is very similar.

3.1 Borel analysis of one loop case (scalar QED)

To begin, we write the one-loop weak-field expansion coefficients $c_n^{(1)}$ in (2.4) in a form which we can easily relate to (3.2). This is clearly already true of the leading behavior (2.11), but we can also quantify the corrections. Recall Euler's formula (1.8) that relates the Bernoulli numbers to the Riemann zeta function. We shift the index of the expansion coefficients so that the weak-field summation in (2.3) begins at $n = 0$, as in (3.1). Thus, we define new expansion coefficients

$$\begin{aligned} a_n^{(1)} \equiv c_{n+2}^{(1)} &= -\frac{\mathcal{B}_{2n+4}}{(2n+2)(2n+4)} \\ &= 2\frac{(-1)^n}{(2\pi)^{2n+4}}\left[\Gamma(2n+3)+\Gamma(2n+2)\right]\left\{1+\frac{1}{2^{2n+4}}+\frac{1}{3^{2n+4}}+\cdots\right\} \end{aligned} \tag{3.8}$$

Keeping the exponentially leading term 1 in the curly bracketed factor, we see that the coefficient $a_n^{(1)}$ is the sum of two terms, each of which has the form in (3.2), with $\mu = 2$,

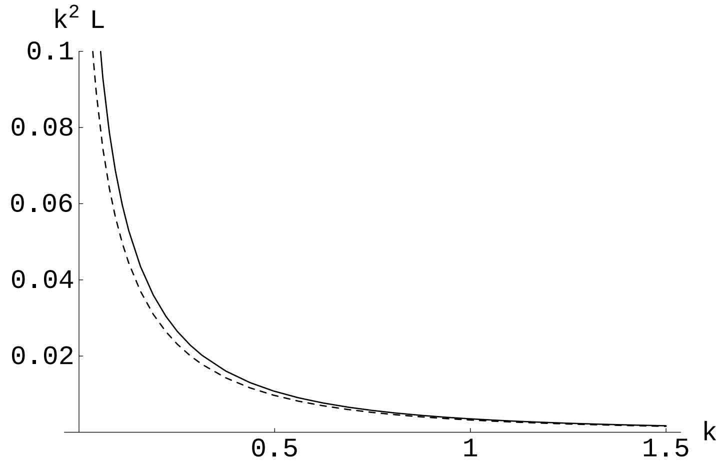

Figure 5: This plot shows the exact one-loop scalar QED effective lagrangian (2.1) [solid curve] in comparison with the leading Borel approximation in (3.9) [dashed curve]. We have multiplied through each function by a common factor of $(4\pi)^2\kappa^2/m^4$ for ease of presentation. Note that the agreement is much better than for the weak-field (large κ) expansion shown in figures 1 and 2.

$\rho = 1/(2\pi)^2$, and $\nu = 3$ and 2 respectively. So, a direct application of the Borel integral expression (3.6) leads to the leading Borel approximation

$$\mathcal{L}^{(1)}_{\text{leading}}(\kappa) = \frac{m^4}{(4\pi)^2}\frac{1}{2\pi^4\kappa^2}\int_0^\infty dt\frac{e^{-2\kappa t}}{1+t^2/\pi^2}\left(t+2\kappa t^2\right) \tag{3.9}$$

Figure 5 shows a comparison of this leading Borel approximation with the exact effective lagrangian (as before, each with an extra factor of κ^2). Note that it does very well at large κ (weak field), but less well at small κ (strong field). Nevertheless, it does much better in the strong field regime than the partial sums of the weak-field expansion (compare with figure 2).

Successive subleading corrections to this leading Borel approximation follow from considering the (exponential) corrections inside the curly brackets in the expression (3.8) for the expansion coefficients. Once again, these terms are of the "standard form" in (3.2), now with $\rho = 1/(2\pi k)^2$, for $k = 2, 3, \ldots$ Thus, each term can be resummed using the Borel formula (3.6). This leads to

$$\mathcal{L}^{(1)}_{\text{borel}}(\kappa) = \frac{m^4}{(4\pi)^2}\frac{1}{2\pi^4\kappa^2}\sum_{k=1}^{\infty}\frac{1}{k^4}\int_0^\infty dt\frac{e^{-2\kappa t}}{1+\frac{t^2}{k^2\pi^2}}\left(t+2\kappa t^2\right) \tag{3.10}$$

Keeping successive terms in the expansion (3.10) rapidly improves the agreement beyond that shown in figure 5 for the leading Borel approximation (3.9).

In fact, the expression (3.10) could have been derived directly by interchanging the summation and integration. To see this, note that by an integration by parts we can rewrite the proper-time form (2.1) of the effective lagrangian as

$$\mathcal{L}^{(1)}(\kappa) = \frac{m^4}{(4\pi)^2}\frac{1}{4\kappa^2}\int_0^\infty dt e^{-2\kappa t}\left(-\coth t+\frac{1}{t}+\frac{t}{3}\right)\left(\frac{1}{t^2}+\frac{2\kappa}{t}\right) \tag{3.11}$$

JHEP06(2002)042

In the large κ limit we can use the small t expansion of the coth function [13]

$$\coth t - \frac{1}{t} - \frac{t}{3} = -\frac{2}{\pi^4}\sum_{k=1}^{\infty}\frac{t^3}{k^4(1+\frac{t^2}{k^2\pi^2})} \tag{3.12}$$

from which the Borel result (3.10) follows. Thus, the proper-time integral representation (2.1) is precisely the Borel integral representation of the divergent asymptotic weak-field series (2.3). And, conversely, the weak-field series expansion (2.3) is just the large κ asymptotic expansion of the proper-time integral expression (2.1). This is completely analogous to what happens in the case of a constant magnetic field background [33].

Now we turn to the Borel analysis of the imaginary part of the effective lagrangian. Recall that in the magnetic/electric case, the transition from a magnetic to an electric background involved the perturbative replacement $B^2 \to -E^2$, or $B \to iE$. This is because the Lorentz invariant combination is $(B^2 - E^2)$. Similarly, in the self-dual case, we change from the self-dual 'magnetic' case to the self-dual 'electric' case by the the analytic continuation $\kappa \to i\kappa$. Under this transformation, the only change in the weak-field expansion (2.3) is that it now becomes a *non-alternating* divergent series. Thus, according to the Borel dispersion relation formula (3.7), the effective lagrangian acquires an imaginary part. Using this Borel dispersion relation formula (3.7), together with the explicit expression (3.8) for the expansion coefficients (but now with the alternating $(-1)^n$ factor suppressed), we find

$$\mathrm{Im}\left[\mathcal{L}^{(1)}(i\kappa)\right] = \frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\sum_{k=1}^{\infty}\left(\frac{2\pi\kappa}{k}+\frac{1}{k^2}\right)e^{-2\pi k\kappa} \tag{3.13}$$

This agrees precisely with what one finds by a direct analytic continuation of the proper-time integral representation (2.1), together with a principal parts prescription for the poles.

3.2 Borel analysis of two-loop case (scalar QED)

The Borel analysis at two-loop is more complicated because the corrections to the leading behavior (2.22) of the two-loop weak-field expansion coefficients (2.16) are considerably more complicated than for the one-loop case (3.8) where the corrections were simple exponentials. Nevertheless, an analysis of the large order behavior of the two-loop coefficients (2.16) uncovers a remarkably rich number theoretic structure, and permits a Borel analysis of the complete prefactor to the leading exponential behavior. This will be shown in this section.

As before, we shift the summation index in the weak-field expansion (2.15) so that it begins at $n = 0$. So we define the expansion coefficients

$$a_n^{(2)} \equiv c_{n+2}^{(2)} \tag{3.14}$$

where $c_n^{(2)}$ are the two-loop weak-field expansion coefficients given by (2.16). In the appendix we show that if we neglect sub-leading exponential corrections (this is analogous to just taking the 1 term in the curly brackets in (3.8) in the one-loop case), then[1] as $n \to \infty$,

$$a_n^{(2)} \sim 2\frac{(-1)^n}{(2\pi)^{2n+4}}\left[\Gamma(2n+3) - \frac{3}{4}\Gamma(2n+2) - \frac{3}{(2\pi)^2}\sum_{l=2}^{n+1}\frac{(-1)^l(2\pi)^{2l}\mathcal{B}_{2l}}{2l}\Gamma(2n+4-2l)\right] \tag{3.15}$$

This should be compared with the analogous formula for the one-loop case when we make the same approximation of ignoring the exponential corrections in (3.8):

$$a_n^{(1)} \sim 2\frac{(-1)^n}{(2\pi)^{2n+4}}\left[\Gamma(2n+3) + \Gamma(2n+2)\right] \tag{3.16}$$

In each case we have written the expansion coefficients as a sum of gamma functions, each term of which has the general form in (3.2). The difference is that at one-loop there are only two terms, while at two-loop there is an infinite series of terms. This translates directly into the fact that in the one-loop imaginary case, the prefactor of the exponential in (3.13) has two terms, while in the two-loop case we will find an infinite series of terms in the prefactor (see (3.20) below).

Given the expansion (3.15) it is straightforward to use the Borel formula (3.6) to write the leading Borel approximate integral representation for the two-loop effective lagrangian:

$$\mathcal{L}^{(2)}_{\rm leading}(\kappa) = \alpha\pi\frac{m^4}{(4\pi)^2}\frac{1}{\pi^4\kappa}\int_0^\infty dt\frac{e^{-2\kappa t}t^2}{1+t^2/\pi^2}\left[1 - \frac{3}{8\kappa t} - \frac{6\kappa t}{(2\pi)^2}\sum_{l=2}^{\infty}\frac{(-1)^l\mathcal{B}_{2l}}{2l(\kappa t/\pi)^{2l}}\right] \tag{3.17}$$

Using the expansion [13]:

$$Re\left[\psi(1+iy)\right] = \ln y + \sum_{l=1}^{\infty}\frac{(-1)^{l-1}\mathcal{B}_{2l}}{2ly^{2l}} \tag{3.18}$$

we can rewrite the leading Borel approximation (3.17) as

$$\mathcal{L}^{(2)}_{\rm leading}(\kappa) = \tag{3.19}$$
$$= \alpha\pi\frac{m^4}{(4\pi)^2}\frac{1}{\pi^4\kappa}\int_0^\infty dt\frac{e^{-2\kappa t}t^2}{1+t^2/\pi^2}\left[1 - \frac{1}{2\kappa t} - \frac{6\kappa t}{(2\pi)^2}\left(\mathrm{Re}\left(\psi(1+\frac{i\kappa t}{\pi})\right) - \ln\left(\frac{\kappa t}{\pi}\right)\right)\right]$$

In figure 6 we compare this leading Borel integral representation (3.19) with the exact two-loop effective lagrangian (1.20). Notice that the agreement is not as good as in the one-loop case (compare with figure 5). This can be traced to the fact that the second term in the expansion (3.15) of the two-loop weak-field expansion coefficients is negative. On the other hand, the second term in the expansion (3.16) of the one-loop weak-field expansion coefficients is positive. This second term, in each case, leads to a log divergence as $\kappa \to 0$. In the one-loop case this log divergence has the same sign as the log divergence of the full

[1]The derivation of this expansion requires some nontrivial results in number theory. This is explained in the appendix.

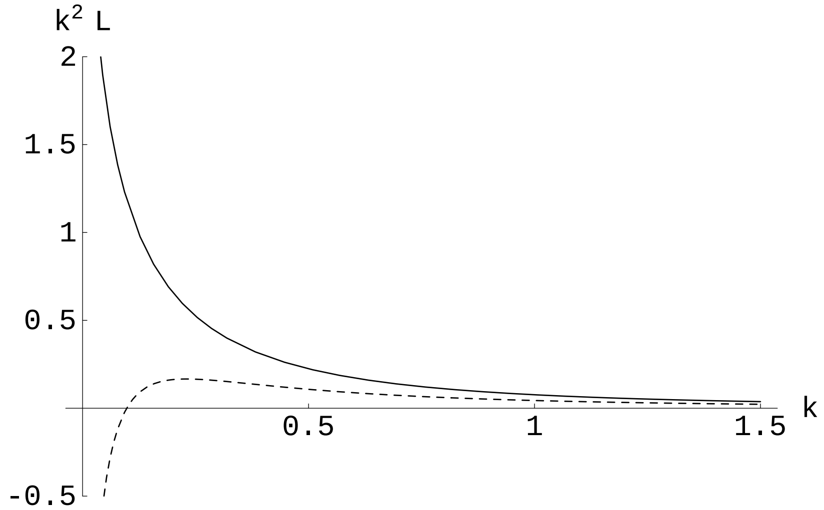

Figure 6: This plot shows the exact two-loop scalar QED effective lagrangian (1.20) [solid curve] in comparison with the leading Borel approximation in (3.19) [dashed curve]. We have multiplied through each function by a common factor of $(4\pi)^2\kappa^2/m^4$ for ease of presentation. Note that the agreement is not as good in the small κ (i.e., strong field) region as that obtained in the one-loop case in figure 5. See text for an explanation.

answer (see (2.10)), but in the two-loop case the log divergence coming from the leading Borel expression (3.19) has the opposite sign compared to the log divergence of the exact answer in the strong field (small κ) limit - compare with (2.21). Hence, this mis-match becomes more dramatic at small κ, as is clearly seen in figure 6.

Now consider the analytic continuation $\kappa \to i\kappa$. The only difference in the weak-field expansion is that the series (2.15) becomes a non-alternating series. Thus, according to the general Borel dispersion relation, there should be a series of imaginary parts, which can be deduced from the formula (3.7). This leads immediately to (here we use a principal parts prescription for the pole at $s = \pi$)

$$\mathrm{Im}\left[\mathcal{L}^{(2)}_{\text{leading}}(i\kappa)\right] = \alpha\pi\frac{m^4}{(4\pi)^2}\frac{1}{2\kappa}\left[1 - \frac{1}{2\pi\kappa} - \frac{3\kappa}{2\pi}\sum_{l=1}^{\infty}\frac{(-1)^l\mathcal{B}_{2l}}{2l\kappa^{2l}}\right]e^{-2\pi\kappa} \tag{3.20}$$

Note that the dominant part, at large κ, of the prefactor (the 1 in the square brackets) is exactly the same as the one-loop leading term in (3.13), except for an overall factor of $\alpha\pi$. In the language of the Borel analysis, this can be traced directly to the fact that the leading growth rates of the one-loop and two-loop expansion coefficients in (2.11) and (2.22), respectively, agree precisely. (The overall factor of $\alpha\pi$ in the two-loop case was explicitly separated out in front of the two-loop weak-field expansion (2.15)). This is the same behavior as is found in the case of a constant electric field background, where the leading two-loop imaginary part is $\alpha\pi$ times the leading one-loop imaginary part [compare Equations (1.4), (1.13) and (1.14)], and which is reflected also in the same leading growth rates of the one- and two-loop weak-field expansion coefficients [21].

We can now compare this with the imaginary part obtained from the exact closed-form expression (1.20). Using the fact that [13]

$$\operatorname{Im}[\psi(i\kappa)] = \frac{1}{2\kappa} + \frac{\pi}{2}\coth(\pi\kappa) \tag{3.21}$$

we deduce that

$$\begin{aligned}
\operatorname{Im}\left[\mathcal{L}^{(2)}(i\kappa)\right] &= \alpha\pi\frac{m^4}{(4\pi)^3}\frac{1}{2\kappa^2}\left[3\kappa^2\operatorname{Re}\left(\tilde{\psi}(i\kappa)\right) - 1 - \kappa\frac{d}{d\kappa}\right](\coth(\pi\kappa)-1) \\
&= \alpha\pi\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\sum_{k=1}^{\infty}\left[3\kappa^2\operatorname{Re}\left(\tilde{\psi}(i\kappa)\right) - 1 + 2\pi\kappa k\right]e^{-2\pi\kappa k} \\
&= \alpha\pi\frac{m^4}{(4\pi)^2}\frac{1}{2\kappa}\sum_{k=1}^{\infty}\left[k - \frac{1}{2\pi\kappa} - \frac{3\kappa}{2\pi}\sum_{l=1}^{\infty}\frac{(-1)^l\mathcal{B}_{2l}}{2l\kappa^{2l}}\right]e^{-2\pi\kappa k}
\end{aligned} \tag{3.22}$$

where we have defined the shorthand function

$$\tilde{\psi}(x) \equiv \psi(x) - \ln x + \frac{1}{2x} \tag{3.23}$$

Thus, the leading Borel expression (3.20) is simply the $k = 1$ term (i.e., the leading exponential contribution) in this expansion. This is completely consistent with our earlier approximation of only keeping power-law (but not exponential) corrections to the expression for the expansion coefficients. So we conclude that the direct use of the Borel dispersion relation (3.7), together with the implied use of a principal parts prescription, is consistent in the two-loop case also.

It is interesting to see that in (3.22) we have the complete Lebedev-Ritus prefactor expansion, for any instanton index k. This should be compared with the two-loop electric field case (1.13) and (1.14), where only the first few terms of this prefactor are known. In particular, it is now clear that this expansion is itself a divergent asymptotic series, a fact which was left undecided by the Lebedev-Ritus analysis. Another interesting issue is the following: comparing the self-dual case result (3.22) with the corresponding one-loop result (3.13) we see that, for any non-zero 'field strength' f, the k-th prefactor at two-loops will dominate over the corresponding one-loop term for sufficiently large k. On the other hand, in the self-dual case, it is straightforward to verify that the *total* two-loop contribution (3.22), when one sums over all k, is always smaller (essentially by a factor of $\alpha\pi$) than the *total* one-loop contribution (3.13). It would be very interesting to see if a similar thing happens in the electric background case, where the individual k terms in the sum have a direct physical meaning in terms of a coherent multi-pair production process [4].

4. Conclusions

To conclude, we have presented an analysis of the imaginary part of the one-loop and two-loop effective Lagangians for scalar QED in a constant self-dual background. At both one- and two-loop, these self-dual effective lagrangians have many properties in common with the effective lagrangians for a constant magnetic or electric field background. They have similar weak- and strong-field expansions. Also, the correspondence between one- and two-loops is very similar in the magnetic/electric and self-dual backgrounds. In particular, in the self-dual case we found that the imaginary part of the two-loop effective lagrangian has a leading term equal to the leading term of the imaginary part of the one-loop effective lagrangian, multiplied by an overall factor of $\alpha\pi$. In terms of the weak-field expansion coefficients, this is reflected in the fact that the one-loop and two-loop expansion coefficients (with a factor of $\alpha\pi$ extracted in the two-loop case) have identical leading growth rates at large order of perturbation theory. We stress that this agreement between leading growth rates is highly nontrivial — it only occurs if one uses the consistently renormalized mass parameter in the expansion, and thus it is sensitive to the finite part of the mass renormalization that enters this analysis at two-loops. These facts also hold for the constant electric field background. It is also consistent with the exponentiation factor, $\exp(\alpha\pi)$, found by Affleck *et al* for the case of pair production in a weak electric field at strong coupling in scalar QED [34]. Indeed, based on the expectation that this exponentiation occurs also in the self-dual case, we conjecture that at loop order l, the weak-field expansion will take the form

$$\mathcal{L}^{(l)(SD)}(\kappa) = \frac{(\alpha\pi)^{l-1}}{(l-1)!}\frac{m^4}{(4\pi)^2}\sum_{n=2}^{\infty} c_n^{(l)}\frac{1}{\kappa^{2n}} \tag{4.1}$$

where for each l, the expansion coefficients $c_n^{(l)}$ have the same *leading* large-N growth rate as $c_n^{(1)}$. Thus, for scalar QED, the $c_n^{(l)}$ should grow factorially exactly as in (2.11), while for spinor QED the *leading* large-N growth should have an additional factor of -2. This also has implications for the leading large-N behavior of the zero momentum limit of the all "+" helicity amplitudes, as is explained in [22] at one- and two-loop. In particular, this conjecture (4.1) for the growth with loop order can be immediately extended to the low energy limit of the 'all +' N-photon amplitude at large N. At first sight this might seem a puzzling result, since it suggests that the N-photon amplitude, if only in one helicity component and in the low energy limit, can be expressed as a *convergent* power series in α for sufficiently large-N. This runs counter to expectations based on well-known general arguments which indicate that the loop expansion should yield an aymptotic series. This apparent contradiction might be resolved either by the appearance of poles in the complex α plane at higher loop orders, invalidating the naive use of the Borel dispersion relations, or, what seems much more likely, by a slowing down of the convergence in N of the ratio $\frac{\Gamma^{(l)(EH)}}{\Gamma^{(1)(EH)}}$ with increasing l. Such a non-uniformity of the convergence might be visible already at the three-loop level, which lends additional motivation to pursuing the calculations presented here to even higher loop order.

We also believe that the results we obtain here for *subleading* behavior, in the self-dual background, can be taken as illustrative of the constant magnetic/electric case also. The advantage, however, of the self-dual case is that there exist closed-form expressions for the two-loop renormalized effective lagrangians. This means that the analysis is much more complete in this case. We were able to find the full imaginary part in the self-dual 'electric' case, and we showed that this is consistent with what one obtains by a Borel dispersion relation treatment of the weak-field expansion, which is a non-alternating divergent series. At two-loop, the prefactor of each exponential term in the imaginary part, has itself an asymptotic expansion.

Acknowledgments

We thank Todd Brun for showing us the identity (A.12), and Richard Stanley for informing us about Miki's identity (A.4). Discussions and correspondence with Vladimir Ritus are also gratefully acknowledged. Both authors thank Dirk Kreimer, and the Center for Mathematical Physics at Boston University, for hospitality during the final stage of this work. We gratefully acknowledge the support of the NSF and CONACyT through a US-Mexico collaborative research grant, NSF-INT-0122615.

A. Asymptotic expansion of two-loop coefficients in scalar QED

In this appendix we compute the large-N–expansion of the coefficients $c_n^{(2)}$, appearing in the two-loop weak field expansion for scalar QED, up to subexponential terms. We recall from (2.15) that the expansion of the final two-loop formula (1.20) yields the following expression for the coefficients:

$$c_n^{(2)} = \frac{1}{(2\pi)^2}\left\{\frac{2n-3}{2n-2}\mathcal{B}_{2n-2} + \frac{3}{2}\sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}}{2k}\frac{\mathcal{B}_{2n-2k}}{(2n-2k)}\right\} \tag{A.1}$$

An alternative form for these expansion coefficients arises when one writes $\mathcal{L}_{\rm scal}^{(1)(SD)}(\kappa)$ as a single proper-time integral (see Equation (3.17) in part I [22]):

$$\begin{aligned} c_n^{(2)} &= \frac{1}{(2\pi)^2}\left\{\frac{2n-3}{2n-2}\mathcal{B}_{2n-2} + 3\left[\psi(2n+1) - \frac{2}{2n-1} + \gamma - 1\right]\frac{\mathcal{B}_{2n}}{2n} + \right. \\ &\qquad \left. + 3\sum_{k=1}^{n-1}\binom{2n-2}{2k-2}\frac{\mathcal{B}_{2k}}{2k}\frac{\mathcal{B}_{2n-2k}}{2n-2k}\right\} \end{aligned} \tag{A.2}$$

This second form is more convenient for studying the large order behaviour of $c_n^{(2)}$. The equivalence of the formulas (A.1) and (A.2) is not immediately obvious. However, it can be shown using Euler's identity for Bernoulli numbers[2]

$$\sum_{k=1}^{n-1}\binom{2n}{2k}\mathcal{B}_{2k}\mathcal{B}_{2n-2k} = -(2n+1)\mathcal{B}_{2n} \tag{A.3}$$

together with a more involved identity known as Miki's identity [37, 38]:

$$\sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}\mathcal{B}_{2n-2k}}{(2k)(2n-2k)} = \sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}\mathcal{B}_{2n-2k}}{(2k)(2n-2k)}\binom{2n}{2k} + \frac{\mathcal{B}_{2n}}{n}\left(\psi(2n+1)+\gamma\right) \tag{A.4}$$

[2]Identities for sums of products of Bernoulli numbers appear in many contexts in physics and mathematics [35, 36].

It is the second form, in (A.2), for $c_n^{(2)}$ which we will use in the following for obtaining the large-N behavior of these coefficients. To start with, we note that the large-N expansion of the first term in the curly brackets is straightforward, using Euler's formula (1.8) which relates the Bernoulli numbers to the zeta function, which is of order 1, with exponentially small corrections. The difficulty is in the last term in (A.2), because the index on each of the Bernoulli numbers need not be large just because n is large. Thus this last term, which involves the finite sum of a product of Bernoulli numbers must be modified somehow to bring it into a more manageable form. We therefore use the Euler identity (A.3) to rewrite (for $n \geq 2$)

$$\begin{aligned}\sum_{k=1}^{n-1}\binom{2n-2}{2k-2}\frac{\mathcal{B}_{2k}}{2k}\frac{\mathcal{B}_{2n-2k}}{2n-2k} &= -\frac{1}{2n(2n-1)}\sum_{k=1}^{n-1}\binom{2n}{2k}\mathcal{B}_{2k}\mathcal{B}_{2n-2k}+ \\ &\quad + 2(-1)^n\frac{(2n-1)!}{(2\pi)^{2n}}\sum_{k=1}^{n-1}\frac{\zeta(2k)\zeta(2n-2k)}{n-k} \\ &= \frac{\mathcal{B}_{2n}(2n+1)}{2n(2n-1)} + 2(-1)^n\frac{(2n-1)!}{(2\pi)^{2n}}\sum_{k=1}^{n-1}\frac{\zeta(2k)\zeta(2n-2k)}{k} \qquad \text{(A.5)}\end{aligned}$$

The advantage of these manipulations is that we can now separate out the leading large-N growth rate of the coefficients and write:

$$\begin{aligned}c_n^{(2)} = (-1)^n\frac{(2n-1)!}{(2\pi)^{2n}}\Bigg\{&\frac{n-3/2}{(n-1)(n-\frac{1}{2})}\zeta(2n-2) - \frac{6}{(2\pi)^2}\left[\psi(2n+1)+\gamma\right]\zeta(2n) + \\ &+ \frac{6}{(2\pi)^2}\sum_{k=1}^{n-1}\frac{1}{k}\zeta(2k)\zeta(2n-2k)\Bigg\} \qquad \text{(A.6)}\end{aligned}$$

In the approximation which neglects exponentially suppressed terms, the large-N behavior of the first two terms is simple, since the ζ - factors can be replaced by unity. The large order behavior of the ψ function follows from using $\psi(m) = \psi(m-1) + \frac{1}{m-1}$, and the asymptotic expansion(2.17). Thus, we find (here and in the following, $\sim$ denotes equality up to terms which are exponentially small for large-N)

$$\begin{aligned}\frac{n-3/2}{(n-1)(n-\frac{1}{2})}\zeta(2n-2) &\sim \frac{1}{n-1} + \sum_{m=2}^{\infty}\frac{(-1)^{m+1}}{2^{m-2}(n-1)^m} \\ \left[\psi(2n+1)+\gamma\right]\zeta(2n) &\sim \log(n-1) + \log(2) + \gamma + \frac{5}{4}\frac{1}{n-1} + \\ &\quad + \sum_{m=2}^{\infty}\frac{1}{(n-1)^m}\left[(-1)^{m+1}(\frac{1}{2}+\frac{1}{2^m}) - \frac{\mathcal{B}_m}{2^m m}\right] \qquad \text{(A.7)}\end{aligned}$$

Note that it will turn out to be convenient to perform the expansion in $n-1$ rather than in n.

The third term in (A.6) is the problematic one. It involves a folded sum of zeta functions. While in many cases such sums can be reexpressed in a form which involves the zeta function only linearly [35, 36], for the case at hand no such formula appears to be known. However, we can still obtain its asymptotic expansion from Euler's formula (A.3), in the following way. Define

$$\begin{aligned} Z_n &\equiv \sum_{k=1}^{n-1} \frac{\zeta(2k)\zeta(2n-2k)}{k} = \sum_{r,s=1}^{\infty} Z_n^{rs} \\ Z_n^{rs} &= \sum_{k=1}^{n-1} \frac{1}{k} r^{-2k} s^{2k-2n} \end{aligned} \tag{A.8}$$

We can immediately discard all terms where not either $r = 1$ or $s = 1$, since they are exponentially small. Thus we have to consider the following three contributions:

1. $r = s = 1$:

$$Z_n^{11} = \sum_{k=1}^{n-1} \frac{1}{k} = \psi(n) + \gamma \tag{A.9}$$

2. $s = 1 < r$:

$$\sum_{r=2}^{\infty} Z_n^{r1} = \sum_{r=2}^{\infty}\sum_{k=1}^{n-1} \frac{1}{kr^{2k}} \sim \sum_{r=2}^{\infty}\sum_{k=1}^{\infty} \frac{1}{kr^{2k}} = -\sum_{r=2}^{\infty} \log\left(1 - \frac{1}{r^2}\right) = \log(2) \tag{A.10}$$

3. $r = 1 < s$:

$$\begin{aligned} \sum_{s=2}^{\infty} Z_n^{1s} &= \sum_{s=2}^{\infty}\sum_{k=1}^{n-1} \frac{1}{ks^{2n-2k}} = \sum_{s=2}^{\infty}\sum_{k=0}^{n-2} \frac{1}{(n-1-k)s^{2k+2}} \\ &= \sum_{m=0}^{\infty} \frac{1}{(n-1)^{m+1}} \sum_{s=2}^{\infty}\sum_{k=0}^{n-2} \frac{k^m}{s^{2k+2}} \sim \sum_{m=0}^{\infty} \frac{1}{(n-1)^{m+1}} \sum_{s=2}^{\infty}\sum_{k=0}^{\infty} \frac{k^m}{s^{2k+2}} \\ &= \sum_{m=0}^{\infty} \frac{1}{(n-1)^{m+1}} \sum_{s=2}^{\infty} \frac{1}{s^2} \left(\frac{1}{s^2}\frac{d}{d(1/s^2)}\right)^m \frac{1}{1-1/s^2} \\ &= \frac{3}{4}\frac{1}{n-1} + \sum_{m=1}^{\infty} \frac{1}{(n-1)^{m+1}} \sum_{k=1}^{m} \mathcal{S}_m^{(k)} k! \sum_{s=2}^{\infty} \frac{1}{(s^2-1)^{k+1}} \end{aligned} \tag{A.11}$$

In the last step we used the combinatorial "normal ordering" identity

$$\left(x\frac{d}{dx}\right)^m = \sum_{k=1}^{m} \mathcal{S}_m^{(k)} x^k \left(\frac{d}{dx}\right)^k \tag{A.12}$$

($m > 0$), where $\mathcal{S}_m^{(k)}$ denotes the Stirling number of the second kind, which is the number of ways of partitioning a set of m elements into k non-empty subsets. Explicitly:

$$\mathcal{S}_m^{(k)} = \sum_{i=0}^{k} (-1)^i \frac{(k-i)^m}{i!(k-i)!} \tag{A.13}$$

JHEP06(2002)042

The s - sum can be done in closed form after using a partial fraction decomposition to write

$$z_k \equiv \sum_{s=2}^{\infty} \frac{1}{(s^2-1)^k} = \frac{(-1)^k}{2^{2k}} \left\{ \sum_{l=1}^{[k/2]} 2^{2l+1} \binom{2k-2l-1}{k-2l} \zeta(2l) - \sum_{l=1}^{k} (1+2^l) \binom{2k-l-1}{k-l} \right\}$$
$$= \frac{(-1)^k}{2^{2k}} \left\{ \sum_{l=1}^{[k/2]} (-1)^{l+1} \frac{2^{2l}(2\pi)^{2l}}{(2l)!} \binom{2k-2l-1}{k-2l} \mathcal{B}_{2l} - 2^{2k-1} \left[1 + \frac{\Gamma(k+1/2)}{\sqrt{\pi} k!} \right] \right\} \quad (A.14)$$

Thus, again using (2.17) we obtain the asymptotic expansion of Z_n:

$$Z_n \sim \log(n-1) + \gamma + \log(2) + \frac{5}{4}\frac{1}{n-1} + \sum_{m=2}^{\infty} \frac{1}{(n-1)^m} \left[\sum_{k=1}^{m-1} k! \mathcal{S}_{m-1}^{(k)} z_{k+1} - \frac{\mathcal{B}_m}{m} \right] \quad (A.15)$$

where the z_k are the numbers defined in (A.14). Finally, putting everything together, the final result for the asymptotic expansion of the coefficients $c_n^{(2)}$ becomes, up to subexponential corrections,

$$c_n^{(2)} \sim (-1)^n \frac{(2n-1)!}{(2\pi)^{2n}} \left\{ \frac{1}{n-1} + \sum_{m=2}^{\infty} \frac{1}{(n-1)^m} \left\{ \frac{(-1)^{m+1}}{2^{m-2}} + \right. \right. \quad (A.16)$$
$$\left. \left. + \frac{6}{(2\pi)^2} \left[(-1)^m \left(\frac{1}{2} + \frac{1}{2^m} \right) + \frac{\mathcal{B}_m}{m} \left(\frac{1}{2^m} - 1 \right) + \sum_{k=1}^{m-1} k! \mathcal{S}_{m-1}^{(k)} z_{k+1} \right] \right\} \right\}$$

Observe that the leading $O(\log(n-1))$ and $O((n-1)^0)$ terms cancel out in the sum. This is a nontrivial self-consistency check. We note that, for the constant term, the occurrence of this cancelation depends on the value of the finite part of the mass shift in the mass renormalization [22]; it happens only if the renormalized scalar mass is the physical one. That the leading asymptotic behaviour depends qualitatively on the finite part of the mass shift term was already observed in the electric field case [21], although then only by numerical means.

In order to be able to do the two-loop Borel analysis, we would like to express these expansion coefficients as a series of Γ function terms. First, rewrite (A.16) in the following form

$$c_n^{(2)} \sim 2\frac{(-1)^n}{(2\pi)^{2n}} \left(\Gamma(2n-1) + \Gamma(2n-2) \right) \left\{ 1 + \sum_{m=1}^{\infty} \frac{d_m}{(n-1)^m} \right\} \quad (A.17)$$

where

$$d_m = \frac{(-1)^m}{2^{m-1}} + \frac{6}{(2\pi)^2} \left[(-1)^{m+1} \left(\frac{1}{2} + \frac{1}{2^{m+1}} \right) + \frac{\mathcal{B}_{m+1}}{m+1} \left(\frac{1}{2^{m+1}} - 1 \right) + \sum_{k=1}^{m} k! \mathcal{S}_m^{(k)} z_{k+1} \right] \quad (A.18)$$

Next, by standard combinatorics one can derive the following rearrangements, which are formally true for any sequence d_m:

$$1 + \sum_{m=1}^{\infty} \frac{d_m}{n^m} = 1 + \frac{e_1}{2n} + \frac{e_2}{2n(2n-1)} + \frac{e_3}{2n(2n-1)(2n-2)} + \cdots,$$
$$e_m = \sum_{k=1}^{m} 2^k S_{m-1}^{(k-1)} d_k \quad (A.19)$$

and

$$1+\sum_{m=1}^{\infty}\frac{d_m}{n^m} = 1+\frac{f_1}{2n-1}+\frac{f_2}{(2n-1)(2n-2)}+\frac{f_3}{(2n-1)(2n-2)(2n-3)}+\cdots\,,$$
$$f_m = \sum_{k=1}^{m} 2^k S_m^{(k)} d_k \tag{A.20}$$

In these formulas, the $S_m^{(k)}$ denote the Stirling numbers of the *first* kind. With the help of these formulas, we can rewrite (A.17) as

$$c_n^{(2)} \sim 2\frac{(-1)^n}{(2\pi)^{2n}}\sum_{m=1}^{\infty}\sigma_{1,m}^{(2)}\Gamma(2n-m)\,, \tag{A.21}$$

where

$$\begin{aligned}\sigma_{1,1}^{(2)} &= 1\,,\\ \sigma_{1,2}^{(2)} &= e_1+1\\ \sigma_{1,m}^{(2)} &= e_{m-1}+f_{m-2}\,, \qquad m\geq 3\end{aligned} \tag{A.22}$$

Using the explicit form (A.18) of d_m we have found, using MATHEMATICA, the following remarkably complex identity (valid for $m \geq 3$):

$$\sum_{k=1}^{m-2}2^k d_k\left(S_{m-2}^{(k-1)}+S_{m-2}^{(k)}\right)+2^{m-1}d_{m-1} = \frac{3(2\pi)^{m-2}}{m}|\mathcal{B}_m| \tag{A.23}$$

Here d_k are defined in (A.18). This identity has the consequence that the expansion coefficients in the decomposition (A.21) are finally very simple:

$$\begin{aligned}\sigma_{1,1}^{(2)} &= 1\\ \sigma_{1,2}^{(2)} &= -\frac{3}{4}\\ \sigma_{1,m}^{(2)} &= 3\frac{(2\pi)^{m-2}}{m}\mid\mathcal{B}_m\mid\,, \qquad m\geq 3\end{aligned} \tag{A.24}$$

In other words, we have shown that, up to exponentially small corrections, as $n\to\infty$,

$$\begin{aligned}c_n^{(2)} &\sim 2\frac{(-1)^n}{(2\pi)^{2n}}\Gamma(2n-1)\left[1-\frac{3}{4(2n-2)}-\frac{3}{(2\pi)^2}\sum_{l=2}^{n-1}\frac{(-1)^l(2\pi)^{2l}\mathcal{B}_{2l}}{2l(2n-2)(2n-3)\cdots(2n-2l)}\right]\\ &= 2\frac{(-1)^n}{(2\pi)^{2n}}\left[\Gamma(2n-1)-\frac{3}{4}\Gamma(2n-2)-\frac{3}{(2\pi)^2}\sum_{l=2}^{n-1}\frac{(-1)^l(2\pi)^{2l}\mathcal{B}_{2l}}{2l}\Gamma(2n-2l)\right]\end{aligned} \tag{A.25}$$

This is the expansion needed to perform the Borel analysis of the two-loop weak field expansion in section 3 : see (3.14) and (3.15).

References

[1] W. Heisenberg and H. Euler, *Folgerungen aus der Diracschen Theorie des Positrons*, *Z. Physik* **C 98** (1936) 714.

[2] V. Weisskopf, *Über die Elektrodynamik des Vakuums auf Grund der Quantentheorie des Elektrons*, Kong. Dans. Vid. Selsk. Math-fys. Medd. XIV No. 6 (1936), reprinted in *Quantum Electrodynamics*, J. Schwinger ed., Dover, New York, 1958.

[3] J. Schwinger, *On gauge invariance and vacuum polarization*, *Phys. Rev.* **82** (1951) 664.

[4] A.I. Nikishov, *Pair production by a constant external field*, *Zh. Eksp. Teor. Fiz.* **57** (1969) 1210, [*Sov. Phys. JETP* **30** (1970) 660];
N.B. Narozhnyi and A.I. Nikishov, *The simplist processes in the pair creating electric field*, *Yad. Fiz.* **11** (1970) 1072 [*Sov. J. Nucl. Phys.* **11** (1970) 596].

[5] S.P. Kim and D.N. Page, *Schwinger pair production via instantons in a strong electric field*, *Phys. Rev.* **D 65** (2002) 105002 [`hep-th/0005078`].

[6] D.L. Burke et al., *Positron production in multiphoton light-by-light scattering*, *Phys. Rev. Lett.* **79** (1997) 1626.

[7] A.C. Melissinos, *The spontaneous breakdown of the vacuum*, `hep-ph/9805507`.

[8] A. Ringwald, *Pair production from vacuum at the focus of an x-ray free electron laser*, *Phys. Lett.* **B 510** (2001) 107 [`hep-ph/0103185`].

[9] R. Alkofer, M.B. Hecht, C.D. Roberts, S.M. Schmidt and D.V. Vinnik, *Pair creation and an x-ray free electron laser*, *Phys. Rev. Lett.* **87** (2001) 193902 [`nucl-th/0108046`].

[10] A.I. Vainshtein, *Decaying systems and divergence of the series of perturbation theory*, Novosibirsk preprint, December 1964. English translation to appear in the Proceedings of QCD 2002/Arkadyfest, M. Shifman ed.

[11] For an extensive review, see J-C. Le Guillou and J. Zinn-Justin, *Large-order behaviour of perturbation theory*, North-Holland, Amsterdam, 1990.

[12] J. Zinn-Justin, *Quantum field theory and critical phenomena*, Oxford University Press, 1997.

[13] M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, Dover, New York, 1972.

[14] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products*, Academic Press, New York, 1972.

[15] V.I. Ritus, *Lagrangian of an intense electromagnetic field and quantum electrodynamics at short distances*, *Zh. Eksp. Teor. Fiz.* **69** (1975) 1517, [*Sov. Phys. JETP* **42** (1975) 774].

[16] V.I. Ritus, *Connection between strong-field quantum electrodynamics with short-distance quantum electrodynamics*, *Zh. Eksp. Teor. Fiz.* **73** (1977) 807, [*Sov. Phys. JETP* **46** (1977) 423].

[17] V.I. Ritus, *The lagrangian function of an intense electromagnetic field*, in *Proc. Lebedev Phys. Inst.* vol. **168**, *Issues in intense-field quantum electrodynamics*, V.I. Ginzburg, ed., Nova Science Pub., NY 1987

[18] V. Fock, *Die Eigenzeit in der klassischen und in der Quantenmechanik*, *Phys. Z. Sowjet.* **12** (1937) 404.

JHEP06(2002)042

[19] S.L. Lebedev and V.I. Ritus, *Virial representation of the imaginary part of the lagrangian function of an electromagnetic field*, *Zh. Eksp. Teor. Fiz.* **86** (1984) 408, [*Sov. Phys. JETP* **59** (1984) 237]

[20] V.I. Ritus, *Method of eigenfunctions and mass operator in quantum electrodynamics of a constant field*, *Zh. Eksp. Teor. Fiz.* **75** (1978) 1560, [*Sov. Phys. JETP* **48** (1978) 788].

[21] G.V. Dunne and C. Schubert, *Two-loop Euler-Heisenberg QED pair-production rate*, *Nucl. Phys.* **B 564** (2000) 591 [hep-th/9907190].

[22] G.V. Dunne and C. Schubert, *Two-loop self-dual Euler-Heisenberg lagrangians 1. Real part and helicity amplitudes*, hep-th/0205004.

[23] G.V. Dunne and C. Schubert, *Closed-form two-loop Euler-Heisenberg lagrangian in a self-dual background*, *Phys. Lett.* **B 526** (2002) 55 [hep-th/0111134].

[24] M.J. Duff and C.J. Isham, *Selfduality, helicity and supersymmetry: the scattering of light by light*, *Phys. Lett.* **B 86** (1979) 157;
M.J. Duff and C.J. Isham, *Selfduality, helicity and coherent states in nonabelian gauge theories*, *Nucl. Phys.* **B 162** (1980) 271.

[25] This function is closely related to Barnes's G-function: E.W. Barnes, *The theory of the G-function*, *Quarterly Journal of Pure and Applied Mathematics* **31** (1900) 264;
E. T. Whittaker and G. N. Watson, *Modern analysis*, Cambridge University Press, 1927 p. 264.

[26] G.V. Dunne, A.W. Thomas and S.V. Wright, *Chiral extrapolation: an analogy with effective field theory*, *Phys. Lett.* **B 531** (2002) 77 [hep-th/0110155].

[27] C.M. Bender and S.A. Orszag, *Advanced mathematical methods for scientists and engineers* Mc Graw-Hill, New York, 1978.

[28] G.B. West, *Asymptotic series and precocious scaling*, *Found. Phys.* **30** (2000) 695–704 [hep-ph/9911416].

[29] F.J. Dyson, *Divergence of perturbation theory in quantum electrodyanmics*, *Phys. Rev.* **85** (1952) 631.

[30] G. Hardy, *Divergent Series*, Oxford University Press, 1949.

[31] G. 't Hooft, *Can we make sense out of 'quantum chromodynamics'?*, in *The Whys of Subnuclear Physics*, A. Zichichi ed., Plenum, New York, 1978. Reprinted in: G. 't Hooft, *Under the spell of the gauge principle*, World Scientific, Singapore, 1994.

[32] M. Beneke, *Renormalons*, *Phys. Rept.* **317** (1999) 1 [hep-ph/9807443].

[33] G.V. Dunne and T.M. Hall, *Borel summation of the derivative expansion and effective actions*, *Phys. Rev.* **D 60** (1999) 065002 [hep-th/9902064].

[34] I.K. Affleck, O. Alvarez and N.S. Manton, *Pair production at strong coupling in weak external fields*, *Nucl. Phys.* **B 197** (1982) 509.

[35] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, math.AG/9810173.

[36] I-C. Huang and S-Y. Huang, *Bernoulli numbers and polynomials via residues*, *J. Number Theor.* **76** (1999) 178.

[37] H. Miki, *A relation between Bernoulli numbers*, *J. Number Theor.* **10** (1978) 297.

[38] I.M. Gessel, *On Miki's identity for Bernoulli numbers*, preprint, Brandeis University, 1999.